

SIMULTANEOUS APPROXIMATION OF A REAL NUMBER BY ALL CONJUGATES OF AN ALGEBRAIC NUMBER

GUILLAUME ALAIN

ABSTRACT. Using a method of H. Davenport and W. M. Schmidt, we show that, for each positive integer n , the ratio $2/n$ is the optimal exponent of simultaneous approximation to real irrational numbers 1) by all conjugates of algebraic numbers of degree n , and 2) by all but one conjugates of algebraic integers of degree $n + 1$.

1. INTRODUCTION

An outstanding problem in Diophantine approximation, motivated initially by Mahler's and Koksma's classifications of numbers, is to provide sharp estimates for the approximation of a real number by algebraic numbers of bounded degree. Starting with the pioneer work [Wi] of E. Wirsing in 1961, this problem has been studied by many authors and extended in several directions. A good account of this can be found in Chapter 3 of [Bu]. For our purpose, let us simply mention that, in 1969, H. Davenport and W. M. Schmidt gave estimates for the approximation by algebraic integers [DS] and that, more recently, D. Roy and M. Waldschmidt looked at simultaneous approximations by several conjugate algebraic integers [RW]. While the latter work was limited to at most one quarter of the conjugates, we consider here the problem of simultaneous approximation of a real number by all (resp. all but one) conjugates of an algebraic number (resp. algebraic integer). Upon defining the *height* $H(P)$ of a polynomial $P \in \mathbb{R}[T]$ to be the largest absolute value of its coefficients, and the *height* $H(\alpha)$ of an algebraic number $\alpha \in \mathbb{C}$ to be the height of its irreducible polynomial in $\mathbb{Z}[T]$, our main result reads as follows.

Theorem A. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and let $n \in \mathbb{N}^*$. There exist positive constants c_1, c_2 depending only on ξ and n with the following properties.*

(i) *There are infinitely many algebraic numbers α of degree n such that*

$$(1) \quad \max_{\bar{\alpha}} |\xi - \bar{\alpha}| \leq c_1 H(\alpha)^{-2/n}$$

where the maximum is taken over all conjugates $\bar{\alpha}$ of α .

(ii) *There are infinitely many algebraic integers α of degree $n + 1$ such that*

$$(2) \quad \max_{\bar{\alpha} \neq \alpha} |\xi - \bar{\alpha}| \leq c_2 H(\alpha)^{-2/n}$$

where the maximum is taken over all conjugates $\bar{\alpha}$ different from α .

1991 *Mathematics Subject Classification.* Primary 11J13; Secondary 11J70.
Work partially supported by NSERC.

In the case $n = 2$, this improves the estimates of the corollary in Section 1 of [AR]. In fact, as we will see in the next section, the statement of part (i) is optimal up to the value of c_1 for each $\xi \in \mathbb{R} \setminus \mathbb{Q}$, while the statement of part (ii) is optimal up to the value of c_2 at least for quadratic irrational values of ξ . This seems to be the first instance where an optimal exponent of approximation is known for all values of the degree n in this type of problem. The fact that we can control the degree of the approximations originates from an observation of Y. Bugeaud and O. Teulié in [BT].

An irrational real number ξ is said to be *badly approximable* if there exists a constant $c > 0$ such that $|\xi - p/q| \geq cq^{-2}$ for any rational number p/q . This is equivalent to asking that ξ has bounded partial quotients in its continued fraction expansion (see Theorem 5F in Chapter 1 of [Sc]). For these numbers, we can refine Theorem A as follows.

Theorem B. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ be badly approximable and let $n \in \mathbb{N}^*$. Then there exist positive constants c_1, \dots, c_4 depending only on ξ and n with the following properties. For each real number $X \geq 1$, there is an algebraic number α of degree n satisfying (1) and $c_3X \leq H(\alpha) \leq c_4X$. There is also an algebraic integer α of degree $n + 1$ satisfying (2) and $c_3X \leq H(\alpha) \leq c_4X$.*

The proof of both results follows the method introduced by Davenport and Schmidt in [DS]. Let $\mathbb{R}[T]_{\leq n}$ denote the real vector space of polynomials of degree $\leq n$ in $\mathbb{R}[T]$, and let $\mathbb{Z}[T]_{\leq n}$ denote the subgroup of polynomials with integral coefficients in $\mathbb{R}[T]_{\leq n}$. We first provide estimates for the last minimum of certain convex bodies of $\mathbb{R}[T]_{\leq n}$ with respect to $\mathbb{Z}[T]_{\leq n}$ and then deduce the existence of polynomials of $\mathbb{Z}[T]_{\leq n}$ with specific inhomogeneous Diophantine properties. This is done in Section 3. In Section 4, we show that these polynomials have roots which fulfil the requirements of Theorem A or B.

Throughout this paper, all implied constants in the Vinogradov symbols \gg , \ll and their conjunction \asymp depend only on ξ and n .

2. OPTIMALITY OF THE EXPONENTS OF APPROXIMATION

Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and let $n \in \mathbb{N}^*$. If $n \geq 2$, the result in part (i) of Theorem A is optimal up to the value of the implied constant since, for any algebraic number α of degree n with conjugates $\alpha_1, \dots, \alpha_n$, the discriminant $D(\alpha)$ of α satisfies

$$|D(\alpha)| \leq H(\alpha)^{2(n-1)} \prod_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|^2 \leq H(\alpha)^{2(n-1)} \left(2 \max_{1 \leq i \leq n} |\xi - \alpha_i| \right)^{n(n-1)}$$

Since $D(\alpha)$ is a non-zero integer, its absolute value is ≥ 1 , and thus we deduce that

$$\max_{1 \leq i \leq n} |\xi - \alpha_i| \geq \frac{1}{2} H(\alpha)^{-2/n}$$

(compare with §5 of [Wi]). If $n = 1$, the result is optimal for any badly approximable ξ . Note that a similar argument also shows that, for any algebraic integer α of degree $n + 1$ with conjugates $\alpha_1, \dots, \alpha_{n+1}$, we have $\max_{1 \leq i \leq n} |\xi - \alpha_i| \geq (1/2) H(\alpha)^{-2/(n-1)}$.

Similarly, the result in part (ii) of Theorem A is optimal up to the value of the implied constant when ξ is a quadratic irrational number. To prove this, suppose that an algebraic integer α of degree $n+1$ has conjugates $\alpha_1, \dots, \alpha_{n+1}$ distinct from ξ with the first n satisfying

$$\max_{1 \leq i \leq n} |\xi - \alpha_i| \leq 1$$

Let $Q(T) \in \mathbb{Z}[T]$ be the irreducible polynomial of ξ over \mathbb{Z} . Since α is an algebraic integer, the product $Q(\alpha_1) \cdots Q(\alpha_{n+1})$ is a rational integer and since it is non-zero (because ξ is not a conjugate of α), we deduce that

$$1 \leq \prod_{i=1}^{n+1} |Q(\alpha_i)|.$$

For each $i = 1, \dots, n$, we have $|Q(\alpha_i)| \ll |\xi - \alpha_i|$ since ξ is a root of Q and $|\xi - \alpha_i| \leq 1$. We also have $|Q(\alpha_{n+1})| \ll \max\{1, |\alpha_{n+1}|\}^2$ since Q has degree 2. This gives

$$1 \ll H(\alpha)^2 \prod_{i=1}^n |\xi - \alpha_i|$$

and consequently $\max_{1 \leq i \leq n} |\xi - \alpha_i| \gg H(\alpha)^{-2/n}$.

Remark 1. It would be interesting to know if there exists as well transcendental numbers ξ for which the exponent $2/n$ for $H(\alpha)$ in Theorem A part (ii) is best possible.

Remark 2. The case where $\xi \in \mathbb{Q}$ is not interesting as it leads to much weaker estimates. In this case, one finds that, for each algebraic number α of degree n with $\alpha \neq \xi$, one has $\max_{\bar{\alpha}} |\xi - \bar{\alpha}| \gg H(\alpha)^{-1/n}$, and that, for each algebraic integer α of degree $n+1$ with $\alpha \neq \xi$, one has $\max_{\bar{\alpha} \neq \alpha} |\xi - \bar{\alpha}| \gg H(\alpha)^{-1/n}$.

3. CONSTRUCTION OF POLYNOMIALS

Throughout this section, we fix an irrational real number $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and a positive integer $n \geq 1$. For each integer $q \geq 1$, we denote by $\mathcal{C}(q)$ the convex body of $\mathbb{R}[T]_{\leq n}$ which consists of all polynomials $P \in \mathbb{R}[T]_{\leq n}$ satisfying

$$|P^{[k]}(\xi)| \leq q^{2k-n} \quad (0 \leq k \leq n)$$

where $P^{[k]}(\xi) = P^{(k)}(\xi)/k!$ denotes the k -th divided derivative of P at ξ (the coefficient of $(T - \xi)^k$ in the Taylor expansion of P at ξ). We first prove :

Proposition 3.1. *Let q be the denominator of a convergent of ξ . Then the last minimum of $\mathcal{C}(q)$ with respect to the lattice $\mathbb{Z}[T]_{\leq n}$ is $\leq 2^n$, and its first minimum is $\geq \left(2^{n^2}(n+1)!\right)^{-1}$. Moreover, the convex body $2^n \mathcal{C}(q)$ contains a basis of $\mathbb{Z}[T]_{\leq n}$ over \mathbb{Z} .*

Proof. Put $L_1 = qT - p$ where p/q denotes a convergent of ξ with denominator q . If $q > 1$, we also define $L_0 = q_0T - p_0$ where p_0/q_0 is the previous convergent of ξ (in reduced form).

If $q = 1$, we simply take $L_0 = 1$. The theory of continued fractions tells us that these linear forms satisfy

$$(3) \quad |L_i(\xi)| \leq q^{-1} \quad \text{and} \quad |L'_i(\xi)| \leq q$$

for $i = 0, 1$, and moreover that their determinant (or Wronskian) is ± 1 (see §4 in Chapter I of [Sc]). The latter fact means that $\{L_0, L_1\}$ spans $\mathbb{Z}[T]_{\leq 1}$ over \mathbb{Z} . Therefore the products $P_j = L_0^j L_1^{n-j}$ ($0 \leq j \leq n$) span $\mathbb{Z}[T]_{\leq n}$ over \mathbb{Z} and, since the rank of $\mathbb{Z}[T]_{\leq n}$ is $n + 1$, they form in fact a basis of $\mathbb{Z}[T]_{\leq n}$ over \mathbb{Z} . Using (3), we also find that

$$|P_j^{[k]}(\xi)| \leq \binom{n}{k} q^{2k-n} \leq 2^n q^{2k-n} \quad (0 \leq j, k \leq n).$$

Thus $\{P_0, \dots, P_n\}$ is a basis of $\mathbb{Z}[T]_{\leq n}$ contained in $2^n \mathcal{C}(q)$. This proves the last assertion of the proposition as well as the fact that the last minimum of $\mathcal{C}(q)$ is $\leq 2^n$.

Identify $\mathbb{R}[T]_{\leq n}$ with \mathbb{R}^{n+1} under the map which sends a polynomial $a_0 + a_1 T + \dots + a_n T^n$ to the point (a_0, a_1, \dots, a_n) . Then the linear map $\theta : \mathbb{R}[T]_{\leq n} \rightarrow \mathbb{R}^{n+1}$ given by $\theta(P) = (P(\xi), P^{[1]}(\xi), \dots, P^{[n]}(\xi))$ has determinant 1 and so $\mathcal{C}(q)$ has volume $\prod_{k=0}^n (2q^{2k-n}) = 2^{n+1}$. Since the lattice $\mathbb{Z}[T]_{\leq n}$ has co-volume 1 (it is identified with \mathbb{Z}^{n+1}), Minkowski's second convex body theorem shows that the successive minima $\lambda_1, \dots, \lambda_{n+1}$ of $\mathcal{C}(q)$ with respect to $\mathbb{Z}[T]_{\leq n}$ satisfy $((n+1)!)^{-1} \leq \lambda_1 \cdots \lambda_{n+1} \leq 1$. Since $\lambda_2 \leq \dots \leq \lambda_{n+1} \leq 2^n$, this implies that $\lambda_1 \geq (2^{n^2}(n+1)!)^{-1}$. \square

The construction of polynomials given by the next proposition uses only the last assertion of Proposition 3.1.

Proposition 3.2. *Let q be the denominator of a convergent of ξ . There exist an irreducible polynomial $P(T) \in \mathbb{Z}[T]$ of degree n and an irreducible monic polynomial $Q(T) \in \mathbb{Z}[T]$ of degree $n + 1$ satisfying*

$$c_5 q^{2k-n} \leq |P^{[k]}(\xi)|, |Q^{[k]}(\xi)| \leq 3c_5 q^{2k-n} \quad (0 \leq k \leq n)$$

where $c_5 = (n+1)2^{n+1}$.

Note that such polynomials have height $\asymp q^n$.

Proof. The last assertion of Proposition 3.1 tells us the existence of a basis $\{P_0, \dots, P_n\}$ of $\mathbb{Z}[T]_{\leq n}$ satisfying

$$(4) \quad |P_j^{[k]}(\xi)| \leq 2^n q^{2k-n} \quad (0 \leq j, k \leq n).$$

Since $\{P_0, \dots, P_n\}$ is a basis of $\mathbb{Z}[T]_{\leq n}$ over \mathbb{Z} , we can write $T^n + 2 = \sum_{j=0}^n b_j P_j(T)$ for some $b_0, \dots, b_n \in \mathbb{Z}$. Consider the polynomial

$$R(T) = 2c_5 \sum_{k=0}^n q^{2k-n} (T - \xi)^k$$

where $c_5 = (n+1)2^{n+1}$. Since $\{P_0, \dots, P_n\}$ is also a basis of $\mathbb{R}[T]_{\leq n}$ over \mathbb{R} , we can also write $R(T) = \sum_{j=0}^n \theta_j P_j(T)$ for some $\theta_0, \dots, \theta_n \in \mathbb{R}$. Choose integers a_0, \dots, a_n such that $a_j \equiv b_j \pmod{4}$ and $|a_j - \theta_j| \leq 2$ for $j = 0, \dots, n$, and define $P(T) = \sum_{j=0}^n a_j P_j(T)$.

By construction $P(T)$ belongs to $\mathbb{Z}[T]_{\leq n}$ and is congruent to $T^n + 2$ modulo 4. Thus it is a polynomial of degree n over \mathbb{Q} and it is irreducible by virtue of Eisenstein's criterion (for the prime 2). Since $P(T) - R(T) = \sum_{j=0}^n (a_j - \theta_j) P_j(T)$, we deduce from (4) that

$$|P^{[k]}(\xi) - R^{[k]}(\xi)| \leq \sum_{j=0}^n |a_j - \theta_j| |P_j^{[k]}(\xi)| \leq c_5 q^{2k-n} \quad (0 \leq k \leq n).$$

Since $R^{[k]}(\xi) = 2c_5 q^{2k-n}$, it follows that $c_5 q^{2k-n} \leq |P^{[k]}(\xi)| \leq 3c_5 q^{2k-n}$ for $k = 0, \dots, n$, as required.

The construction of $Q(T)$ is similar. Write

$$T^{n+1} + 2 = T^{n+1} + \sum_{j=0}^n b'_j P_j(T) \quad \text{and} \quad (T - \xi)^{n+1} + R(T) = T^{n+1} + \sum_{j=0}^n \theta'_j P_j(T),$$

with $b'_0, \dots, b'_n \in \mathbb{Z}$ and $\theta'_0, \dots, \theta'_n \in \mathbb{R}$, and choose integers a'_0, \dots, a'_n such that $a'_j \equiv b'_j \pmod{4}$ and $|a'_j - \theta'_j| \leq 2$ for $j = 0, \dots, n$. Then the polynomial

$$Q(T) = T^{n+1} + \sum_{j=0}^n a'_j P_j(T) \in \mathbb{Z}[T]$$

is irreducible (by virtue of Eisenstein's criterion for 2), monic of degree $n+1$, and it satisfies also $|Q^{[k]}(\xi) - R^{[k]}(\xi)| \leq c_5 q^{2k-n}$ for $k = 0, \dots, n$. \square

4. PROOF OF THEOREMS A AND B

In this section, we prove the main theorems A and B of the introduction by combining Proposition 3.2 with the following result.

Proposition 4.1. *Let $\xi \in \mathbb{R}$, let $n \in \mathbb{N}^*$, let $\delta > 0$ and let \mathcal{P} be a subset of $\mathbb{Z}[T]$. Suppose that the elements of \mathcal{P} are either polynomials of degree n or monic polynomials of degree $n+1$. Then the following conditions are equivalent :*

- (i) *There exists a constant $c_6 > 0$ such that $|P^{[k]}(\xi)| \leq c_6 H(P)^{1-(n-k)\delta}$ for each $P \in \mathcal{P}$ and each $k = 0, 1, \dots, n$.*
- (ii) *There exists a constant $c_7 > 0$ such that $|\xi - \alpha| \leq c_7 H(P)^{-\delta}$ for each $P \in \mathcal{P}$ and for n of the roots α of P , counting multiplicity.*

Proof. Fix $P \in \mathcal{P}$ and write it in the form

$$P(T) = a_0(T - \alpha_1) \cdots (T - \alpha_m)$$

where $m = \deg P$ and $\alpha_1, \dots, \alpha_m$ are the roots of P ordered so that $|\xi - \alpha_1| \leq \dots \leq |\xi - \alpha_m|$. We put $\varepsilon = H(P)^{-\delta}$ and consider the polynomial

$$R(T) = P(\varepsilon T + \xi) = a_0 \varepsilon^m \prod_{k=1}^m (T + \varepsilon^{-1}(\xi - \alpha_k)).$$

The height of R is

$$H(R) = \max_{0 \leq k \leq m} |R^{[k]}(0)| = \max_{0 \leq k \leq m} |P^{[k]}(\xi)| \varepsilon^k,$$

and its Mahler measure is

$$M(R) = |a_0| \varepsilon^m \prod_{k=1}^m \max\{1, \varepsilon^{-1}|\xi - \alpha_k|\} = |a_0| \prod_{k=1}^m \max\{\varepsilon, |\xi - \alpha_k|\}.$$

For convenience, we also define

$$L = \begin{cases} |a_0| & \text{if } m = n \\ \max\{\varepsilon, |\xi - \alpha_m|\} & \text{if } m = n + 1 \end{cases}$$

so that the formula for $M(R)$ becomes

$$M(R) = L \prod_{k=1}^n \max\{\varepsilon, |\xi - \alpha_k|\}$$

(recall that $a_0 = 1$ when $m = n + 1$). Our argument below is based on the standard inequalities relating these notions of heights, namely

$$M(R) \leq (m + 1)H(R) \quad \text{and} \quad H(R) \leq 2^m M(R).$$

If condition (ii) holds, we find that $M(R) \leq c_7^n \varepsilon^n L$. We also have $L \ll H(P)$ since $|a_0| \leq H(P)$ and since $|\xi - \alpha| \ll \max\{1, |\alpha|\} \ll H(P)$ for any root α of P . Then, for each $k = 0, \dots, n$, we obtain

$$|P^{[k]}(\xi)| \ll \varepsilon^{-k} H(R) \ll \varepsilon^{-k} M(R) \ll \varepsilon^{n-k} H(P)$$

which shows that condition (i) holds.

Conversely assume that condition (i) holds. In this case we find that $H(R) \leq c_6 \varepsilon^n H(P)$. We claim that $H(P) \ll L$. If we take this for granted, we deduce that

$$L \varepsilon^{n-1} |\xi - \alpha_n| \leq M(R) \ll H(R) \ll \varepsilon^n L$$

which implies that condition (ii) holds.

To prove the claim, we observe that

$$H(P) \asymp H(P(T + \xi)) = \max_{0 \leq k \leq m} |P^{[k]}(\xi)|.$$

By hypothesis, we have $|P^{[k]}(\xi)| \leq c_6 H(P)^{1-\delta}$ for $k = 0, \dots, n-1$ and we also have $|P^{[m]}(\xi)| = 1$ if $m = n + 1$. Finally, we have $|P^{[n]}(\xi)| = |a_0|$ if $m = n$, and $|P^{[n]}(\xi)| = |\sum_{k=1}^m (\xi - \alpha_k)| \leq m|\xi - \alpha_m|$ if $m = n + 1$, showing that $|P^{[n]}(\xi)| \ll L$. All this implies that

$$H(P) \ll \max\{1, L\}.$$

Since $L \geq \varepsilon = H(P)^{-\delta}$, this in turn implies that $H(P) \ll L$. □

Proof of the theorems. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}^*$. We simply prove Part (ii) of Theorems A and B since the proof of Part (i) is similar and slightly easier.

For each denominator q of a convergent of ξ , Proposition 3.2 shows the existence of an irreducible monic polynomial $Q \in \mathbb{Z}[T]$ of degree $n + 1$ satisfying $H(Q) \asymp q^n$ and

$$|Q^{[k]}(\xi)| \leq c_6 H(Q)^{(2k-n)/n} = c_6 H(Q)^{1-(n-k)(2/n)}, \quad (0 \leq k \leq n)$$

for some constant $c_6 = c_6(\xi, n)$. The family \mathcal{P} of these polynomials satisfies the condition (i) of Proposition 4.1 for the choice $\delta = 2/n$, and so it satisfies also the condition (ii) of the same proposition for the same value of δ and for some constant c_7 . For each $Q \in \mathcal{P}$, choose a root α of Q for which $|\xi - \alpha|$ is maximal. Since Q is irreducible, this root α is an algebraic integer of degree $n + 1$ and height $H(\alpha) = H(Q)$ whose conjugates $\bar{\alpha}$ over \mathbb{Q} are the $n + 1$ distinct roots of Q . Therefore, we get $\max_{\bar{\alpha} \neq \alpha} |\xi - \bar{\alpha}| \leq c_7 H(\alpha)^{-2/n}$. This proves Part (ii) of Theorem A since we find infinitely many such numbers α by varying Q .

If ξ is badly approximable, the ratios of the denominators of consecutive convergents of ξ are bounded. Thus, for each $X \geq 1$, there exists such a denominator q with $q \asymp X^{1/n}$, and so there exists a polynomial $Q \in \mathcal{P}$ with $H(Q) \asymp X$. Consequently, the root α of Q that we chose above satisfies $H(\alpha) \asymp X$ and this proves Part (ii) of Theorem B. \square

Acknowledgments. The author thanks his MSc thesis supervisor Damien Roy for suggesting this problem and for his help in writing the present paper.

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DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'OTTAWA, 585 KING EDWARD, OTTAWA, ONTARIO K1N 6N5, CANADA

E-mail address: gyomalin@gmail.com